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Random Approximations and Fixed Point Theorems

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As an application of our random variational inequality proved in this paper, the random versions of Fan's best approximation theorem for both single-valued and set-valued mappings are given. These results, in turn, are used to derive some random fixed point theorems for both nonself single-valued and set-valued mappings in normed spaces. Our random fixed point theorems are closely related to, but not comparable with results given by Engl, Lin, Reich, Seghal and Waters, Tan and Yuan, and Xu. © 1996 Academic Press, Inc.

1. INTRODUCTION

Random fixed-point theory has received much attention in recent years, for instance, see Bharucha-Reid [2], Bocsan [3], Chang [4], Engl [7], Itoh [11], Kucia and Nowak [12], Lin [15], Liu and Chen [18], Papageorgiou [19], Reich [21], Rybinski [26], Sehgal and Singh [29], Seghal and Waters [30, 31], Tan and Yuan [33, 34] and Xu [37]. In this paper, we consider a stochastic version of the best approximation theorem, i.e., Theorem 2 of Fan in [8] which is stated as follows:

Let X be a nonempty compact convex subset of normed space E. For any continuous mapping f from X to E, there exists a point $u \in X$ such that

$$\|u - f(u)\| = d(f(u); X).$$

The Fan's best approximation theorem is very useful in the study of fixed point theory in topological vector spaces, and this idea has been further illustrated by Reich in [20]. Since then various aspects (non-stochastic versions) of Fan's approximation theorem have been studied by Ding and

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Tan [6], Fan [9], Lin [14], Lin [16], Lin and Yen [17], Reich [22], Singh and Watson [32] and Yuan [38] under different assumptions. Recently, Sehgal and Singh [29], Seghal and Waters [30], Papageorgiou [19], Lin [15] and Tan and Yuan [34] have studied the random version of Fan's best approximation theorem, which is illustrated as follows:

Let X be a non-empty subset of a normed space $(E, \|\cdot\|)$, (Ω, Σ) a measurable space and $f\colon Q \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ an extended real-valued function. To find a single-valued measurable mapping $\psi\colon \Omega \to X$ such that for each $\omega \in \Omega$,

$$\|f(\omega, \psi(\omega)) - \psi(\omega)\| = \inf_{x \in X} \|x - f(\omega, \psi(\omega))\|.$$

It is hoped that the random best approximation theorem will play a role analogous to the role of the best approximation theorem of Fan in the study of deterministic fixed point theory in topological vector spaces. Some results have already been achieved in this line, for example, see Lin [15], Papageorgiou [19], Sehgal and Singh [29], Seghal and Waters [30] and Tan and Yuan [33-34]. Unfortunately, there is no literature for the study of random best approximation theorems through the point of view of random variational inequalities. Our objective in this paper is to study random variational inequalities, and to derive some random best approximation theorems which, in turn, imply random fixed point theorems for both single and set-valued mappings.

The idea used in this paper is illustrated as follows: In order to derive random fixed point theorems, we reduce the existence problem of random fixed points to the existence problem of the random best approximation. The latter can be dealt with using random variational inequalities.

We would like to point out that the method used in this paper is different from those used by Engl [7], Lin [15], Papageorgiou [19], Reich [21], Sehgal and Singh [29], Sehgal and Waters [30], Tan and Yuan [34] and Xu [37] (we just mention a few names here), and our results are closely to related to, but not comparable to, those results given by those authors. In order to present our exposition clearly, we first study single-valued mappings, then the similar idea is applied for the study of set-valued mappings.

Now we introduce some notations and definitions. Let X be a non-empty set. We denote by $\mathscr{F}(X)$ the family of all non-empty finite subsets of X and by 2^X the family of all non-empty subsets of X. Let X be a non-empty subset of topological space E. Then (i) we denote by $\partial_E X$ and $int_E X$ the boundary and relative interior of X in E respectively, which are further simplified to ∂X and $intX$ if there is no confusion; (ii) we denote by X_E^c (simply, by X^c) the complement of X in E, i.e., $X_E^c\colon = \{x \in E\colon x \notin X\}$; and (iii) the convex hull of X in E is denoted by coX. We denote $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integer and the real line, respectively. Let X and Y be topological spaces and $F\colon X \to 2^Y$ be a set-valued mapping. Then (1) F is said to be lower (resp., upper) semicontinuous if for each closed (resp., open) subset C of Y, the set $\{x \in X: F(x) \subset C\}$ is closed (resp., open) in X; and (2) F is continuous if F is both lower and upper semicontinuous. Throughout this paper the terminology map always means a mapping which is single-valued.

A measurable space (Ω, Σ) is a pair where Ω is a set and Σ is a σ-algebra of subsets of Ω. If X is a topological space, the Borel σ-algebra $\mathscr{B}(X)$ is the smallest σ-algebra containing all open subsets of X. If (Ω_1, Σ_1) and (Ω_2, Σ_2) are two measurable spaces, the space $(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2)$ denotes the smallest σ-algebra which contains all the sets of $A \times B$, where $A \in \Sigma_1, B \in \Sigma_2$. We note that the Borel σ-algebra $\mathscr{B}(X_1 \times X_2)$ contains $\mathscr{B}(X_1) \otimes \mathscr{B}(X_2)$ in general.

A map $f: \Omega_1 \to \Omega_2$ is said to be (Σ_1, Σ_2) measurable if for each $B \in \Sigma_2, f^{-1}(B) = \{x \in \Omega_1, f(x) \in B\} \in \Sigma_1$. Let X be a topological space and $F: (\Omega, \Sigma) \to 2^X$ be a correspondence (or say, mapping). Then (a) F is said to be measurable (resp., weakly measurable) if $F^{-1}(B) := \{\omega \in \Omega: F(\omega) \cap B \neq \emptyset\} \in \Sigma$ for each closed (resp., open) subset B of X; (b) the graph of F is said to be measurable if the set $GraphF := \{(\omega, y) \in \Omega \times X: y \in F(\omega)\} \in \Sigma \otimes \mathscr{B}(X)$ and (c) a (single-valued) map $f: \Omega \to X$ is said to be a measurable selection of F provided that f is measurable with $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

Let (X_1, Σ_1), (X_2, Σ_2) be measurable spaces, Y a topological space. Then a mapping $F: X_1 \times X_2 \to 2^Y$ is said to be jointly measurable (resp., jointly weakly measurable) if for every closed (resp., open) subset B of Y, $F^{-1}(B) \in \Sigma_1 \otimes \Sigma_1$. If X is a topological space, then it is understood that Σ is the Borel σ-algebra $\mathscr{B}(X)$.

Let X and Y be two topological spaces, (Ω, Σ) a measurable space and $F: \Omega \times X \to 2^Y$ a mapping. Then (i) F is a random mapping if for each fixed $x \in X$, the mapping $F(\cdot, x): \Omega \to 2^Y$ is measurable; (ii) F is randomly continuous if for each fixed $\omega \in \Omega$, $F(\omega, \cdot): X \to 2^Y$ is continuous and for each fixed $x \in X$, $F(\cdot, x): \Omega \to 2^Y$ is measurable.

A topological space X is (i) a Polish space if X is separable and metrizable under a complete metric; (ii) a Suslin space if X is a Hausdorff topological space and the continuous image of a Polish space. A Suslin (resp., Polish) subset in a topological space is a subset which is a Suslin (resp., Polish) space. The Suslin set plays very important roles in measurable selection theory (for details, see [1], [3], [19], [36]). We note that if X_1 and X_2 are Suslin spaces, then $\mathscr{B}(X_1 \times X_2) = \mathscr{B}(X_1) \otimes \mathscr{B}(X_2)$(e.g., see [27, p. 113]).

Denote by μ and v the sets of infinite and finite sequences of positive integers respectively, let ξ be a family of sets and $F: v \to \xi$ be a map. For each $\sigma = (\sigma_i)_{i=1}^{\infty} \in \mu$ and $n \in \mathbb{N}$, we shall denote $(\sigma_1, ..., \sigma_n)$ by σ/n. Then $\bigcap_{\sigma \in \mu} \bigcup_{n=1}^{\infty} F(\sigma/n)$ is said to be obtained from ξ by the Suslin operation.

Now if every set obtained from ξ in this way is also in ξ, then ξ is called a Suslin family. Note that, if μ is an outer measure on a measurable space (Σ, Ω), then Σ is a Suslin family (see [28, p. 50]). In particular, if (Ω, Σ) is a complete measurable space, then Σ is a Suslin family.

Let (Ω, Σ) be a measurable space, X be a topological space and $F: \Omega \times X \to 2^X$ be a mapping. Then a (single-valued) map $\phi: \Omega \to X$ is said to be a random fixed point of F if ϕ is a measurable map and $\phi(\omega) \in F(\omega, \phi(\omega))$ for all $\omega \in \Omega$. It should be noted that some authors defined a random fixed point of F to be a single-valued measurable mapping ϕ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ for almost every $\omega \in \Omega$, e.g., see [19] and [26]. We also remark that if $F: \Omega \times X \to 2^X$ has a random fixed point, then for each fixed $\omega \in \Omega$, $F(\omega, \cdot)$ has a fixed point in X. However the converse is not true in general (e.g., see examples in [31] and [33]).

We first need the following measurable selection Theorem which is a corollary of Theorem 7 in [13]:

Theorem A. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a Suslin space. Suppose $F: (\Omega, \Sigma) \to 2^X$ is a mapping with $\mathrm{Graph} F \in \Sigma \otimes \mathscr{B}(X)$. Then F has a measurable family $\{g_i(\omega)\}_{i=1}^{\infty}$ of selections such that for each $\omega \in \Omega$, the set $\{g_i(\omega)\}_{i=1}^{\infty}$ is dense in $F(\omega)$. In particular, F has a measurable selection.*

Let X be a non-empty subset of E and $f: \Omega \times X \times X \to \bigcap \{-\infty, +\infty\}$ be a function, where (Ω, Σ) is a measurable space. Then a measurable (single-valued) map $g: \Omega \to X$ is said to be a random variational solution for the *random variational problem* f provided that $\sup_{y \in X} f(\omega, g(\omega), y) \leqslant 0$ for all $\omega \in \Omega$. It is clear that if f has a random variational solution, then for each fixed $\omega \in \Omega$, the operator $f(\omega, \cdot, \cdot)$ has at least one variational solution. The following example illustrates the converse does not hold if the function F lacks the measurable property.

Example. Let $\Omega = X = [0, 1]$, Σ be the σ-algebra of Lebesgue measurable subsets of [0,1] and A be a non-Lebesgue measurable subset of [0,1] (for the existence, see Royden [25, p. 63]). Define $f: \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$f(\omega, x, y) = \begin{cases} (x-1) \cdot y, & \text{if } (\omega, x, y) \in A \times X \times X; \\ x \cdot y, & \text{otherwise.} \end{cases}$$

Then for each fixed $\omega \in \Omega$, $f(\omega, \cdot, \cdot)$ has a unique variational solution $\psi: \Omega \to \mathbb{R}$ defined by

$$\psi(\omega) = \begin{cases} \{1\}, & \text{if } \omega \in A; \\ \{0\}, & \text{otherwise.} \end{cases}$$

However, f does not have any random variational solution as ψ is not measurable.

2. A Random Variational Inequality and Best Approximations

In this section, we first prove the existence of random variational solutions for the function f which has certain measurable properties. This random variational inequality is then applied to give the random approximation theorems for both single and set-valued mappings in normed spaces, which correspond to the deterministic results given by Fan in [9] and [8] and Reich in [20].

THEOREM 2.1. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a nonempty convex Suslin subset of a topological vector space E. Suppose $f: \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is such that*

(a) *for each fixed $(\omega, y) \in \Omega \times X$, $x \to f(\omega, x, y)$ is lower semicontinuous in each nonempty compact subset C of X;*

(b) *for each fixed $(\omega, x) \in \Omega \times X$, $y \to f(\omega, x, y)$ is lower semicontinuous in X;*

(c) *for each fixed $\omega \in \Omega$, for any $A \in \mathcal{F}(X)$ and $x \in co(A)$, $\inf_{y \in A} f(\omega, x, y) \leqslant 0$;*

(d) *there exist a non-empty compact subset K of X and a non-empty convex compact subset X_0 of X such that for each $x \in X \backslash K$, there exists $y \in co(X_0 \cup \{x\})$ with $f(\omega, x, y) > 0$ for all $\omega \in \Omega$;*

(e) *for each fixed $y \in X$, the function $f_y: \Omega \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by $f_y(\omega, x) := f(\omega, x, y)$ for each $(\omega, x) \in \Omega \times X$ is jointly measurable.*

Then there exists a countable measurable family $\{g_i\}_{i=1}^{\infty}$ from Ω to K such that for each g_i,

$$\sup_{y \in X} f(\omega, g_i(\omega), y) \leqslant 0$$

for all $\omega \in \Omega$.

Proof. Define $\phi: \Omega \to 2^K$ by

$$\phi(\omega) := \{x \in K: \sup_{y \in X} f(\omega, x, y) \leqslant 0\}$$

for each $\omega \in \Omega$. Then for each fixed $\omega \in \Omega$, our conditions (a)–(d) imply that the function $f(\omega, \cdot, \cdot)$ satisfies all hypotheses of Theorem 1 of Ding and Tan in [5]. By Theorem 1 of [5], $\phi(\omega)$ is a non-empty and closed subset of K. Let B be a countable dense subset of X, where X is a Suslin set. From the lower semicontinuity of $y \mapsto f(\omega, x, y)$ for each fixed $(\omega, x) \in \Omega \times X$, we have,

$$\begin{aligned}\phi(\omega) &= \{x \in K: \sup_{y \in X} f(\omega, x, y) \leqslant 0\} \\ &= \bigcap_{y \in X} \{x \in X: f(\omega, x, y) \leqslant 0\} \\ &= \bigcap_{y_i \in B} \phi_i(\omega),\end{aligned}$$

where $\phi_i(\omega) := \{x \in X: f(\omega, x, y_i) \leqslant 0\}$. From the condition (e), *Graph* $\phi_i = f_{y_i}^{-1}((-\infty, 0]) \in \Sigma \otimes \mathscr{B}(X)$. Therefore *Graph* $\phi \in \Sigma \otimes \mathscr{B}(X)$. Now by measurable selection Theorem A, there exists a countable measurable family $\{g_i\}_{i=1}^{\infty}$ from Ω to K such that $\overline{\{g_i(\omega): i = 1, ..., \}} = \phi(\omega)$ for each $\omega \in \Omega$. Thus for each g_i,

$$\sup_{y \in X} f(\omega, g_i(\omega), y) \leqslant 0$$

for all $\omega \in \Omega$ and the conclusion follows. ∎

Let X and Y be non-empty subsets of a normed space $(E, \|\cdot\|)$. For each point $u \in E$, we denote by $d(u, X) := \inf_{x \in X} \|u - x\|$ the distance between u and X, and $d(A, B) := \inf_{x \in A} d(x, B)$ the distance between A and B.

As an application of Theorem 2.1, we have the following stochastic approximation result which corresponds to the well-known best approximation theorem of Fan in [8] (see also Fan [9], Reich [20] and Yuan [38]):

THEOREM 2.2. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty convex Suslin subset of a normed space $(E, \|\cdot\|)$. Suppose $\psi: \Omega \times X \to E$ is a randomly continuous map, and moreover there exist a non-empty convex compact subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \backslash K$, there exists $x \in X_0$ with $\|x - \psi(\omega, y)\| < \|y - \psi(\omega, y)\|$ for all $\omega \in \Omega$. Then there is a countable measurable family $\{g_i\}_{i=1}^{\infty}$ from Ω to K such that for each g_i,*

$$\|g_i(\omega) - \psi(\omega, g_i(\omega))\| = d(X, \psi(\omega, g_i(\omega)))$$

for all $\omega \in \Omega$.

Proof. Define a function $f: \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$f(\omega, x, y) := \|x - \psi(\omega, x)\| - \|y - \psi(\omega, x)\|$$

for each $(\omega, x, y) \in \Omega \times X \times X$. Then f satisfies all hypotheses of Theorem 2.1. By Theorem 2.1, there exists a countable measurable family $\{g_i\}_{i=1}^{\infty}$

from Ω to K such that for each g_i and $\omega \in \Omega$, $\|g_i(\omega) - \psi(\omega, g_i(\omega))\| = \inf_{x \in X} \|x - \psi(\omega, g_i(\omega))\|$ and the proof is complete. ∎

Remark 1. Theorem 2.2 is closely related to, but not comparable to, Theorem 1 of Lin [15], Theorem 1 of Sehgal and Waters [31].

As an immediate application of Theorem 2.2, we have the following random best approximation in the compact setting:

COROLLARY 2.3. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty convex Suslin set in a normed space $(E, \|\cdot\|)$. Suppose $\psi: \Omega \times X \to E$ is a randomly continuous map. Then there is a countable measurable family $\{g_i\}_{i=1}^{\infty}$ from Ω to X such that for each g_i,*

$$\|g_i(\omega) - \psi(\omega, g_i(\omega))\| = d(X, \psi(\omega, g_i(\omega)))$$

for all $\omega \in \Omega$.

Proof. Note that X is non-empty convex compact so that X is Suslin. Therefore the conditions of Theorem 2.2 are satisfied automatically and the conclusion of follows. ∎

As another application of Theorem 2.1, we have the following stochastic approximation theorem for set-valued mappings:

THEOREM 2.4. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty convex Suslin subset of a normed space $(E, \|\cdot\|)$. Suppose $\psi: \Omega \times X \to 2^E$ is randomly continuous mapping with non-empty compact and convex values. Moreover suppose there exist a non-empty convex compact subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \backslash K$, there exists $x \in X_0$ with $\inf_{u \in \psi(\omega, y)} \|x - u\| < \inf_{u \in \psi(\omega, y)} \|y - u\|$ for all $\omega \in \Omega$.*

Then there is a countable measurable family $\{g_i\}_{i=1}^{\infty}$ from Ω to K such that for each g_i,

$$\inf_{u \in \psi(\omega, g_i(\omega))} \|g_i(\omega) - u\| = d(X, \psi(\omega, g_i(\omega)))$$

for all $\omega \in \Omega$.

Proof. Following the argument similar to that used in the proof of Theorem 2.2, we define $f: \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$f(\omega, x, y) = \inf_{z \in \psi(\omega, x)} \|z - x\| - \inf_{z \in \psi(\omega, x)} \|z - y\|.$$

for each $(\omega, x, y) \in \Omega \times X \times X$. Note that $\psi(\Omega, x)$ is non-empty compact, the mapping $(\omega, x, y) \mapsto f(\omega, x, y)$ is randomly continuous by Lemma 3 of

Sehgal and Singh [29]. Now we show that the function f satisfies all conditions of Theorem 2.1. Fixing each $\omega \in \Omega$. For each $A \in \mathcal{F}(X)$ and each $x \in co(A)$, we must have $\min_{y \in A} f(\omega, x, y) \leqslant 0$. Otherwise there exist $A := \{y_i, ..., y_n\} \in \mathcal{F}(X)$ and $x = \sum_{i=1}^n \lambda_i y_i \in co(A)$ with $\lambda_i, ..., \lambda_n \geqslant 0$ and $\sum_{i=1}^n \lambda_i$ such that $f(\omega, x, y_i) > 0$ for all $i = 1, ..., n$. Since $\psi(\omega, x)$ is compact, there exists $z_i \in \psi(\omega, x)$ such that $\|z_i - y_i\| = \inf_{z \in \psi(\omega, x)} \|z - y_i\|$ for $i = 1, 2, ..., n$, i.e.,

$$\begin{aligned} f(\omega, x, y_i) &= \inf_{z \in \psi(\omega, x)} \|z - x\| - \inf_{z \in \psi(\omega, x)} \|z - y_i\| \\ &= \inf_{z \in \psi(\omega, x)} \|z - x\| - \|z_i - y_i\| \end{aligned}$$

for each $i = 1, ..., n$. Let $z_0 = \sum_{i=1}^n \lambda_i z_i$, then $z_0 \in \psi(\omega, x)$ as $F(\omega, x)$ is convex. It follows that

$$\begin{aligned} 0 < f(\omega, x, y_i) &= \inf_{z \in \psi(\omega, x)} \|z - x\| - \inf_{z \in \psi(\omega, x)} \|z - y_i\| \\ &\leqslant \|z_0 - x\| - \inf_{z \in \psi(\omega, x)} \| - y_i\| \\ &\leqslant \sum_{i=1}^n \lambda_i \|z_i - y_i\| - \inf_{z \in \psi(\omega, x)} \|z - y_i\| = 0, \end{aligned}$$

which is impossible. Thus f satisfies all conditions of Theorem 2.1. By Theorem 2.1, there exists a countable measurable maps $\{g_i\}_{i \in \mathbb{N}}$ from Ω to K such that each g_i, we have $\sup_{y \in X} f(\omega, \psi_i(\omega), y) \leqslant 0$ for all $\omega \in \Omega$. Hence

$$d(\psi(\omega, g_i(\omega)), g_i(\omega)) = d(\psi(\omega, g_i(\omega)), X)$$

for all $\omega \in \Omega$ and we complete the proof. ∎

For other kinds of random approximations and their applications to the study of random fixed points, the interested reader is referred to Lin [15], Sehgal and Singh [29], and Tan and Yuan [34]. In addition, some applications of random variational inequalities to stochastic partial differential equations have also been given by Tan, Tarafdar and Yuan in [35].

3. Random Fixed Point Theorems for Nonself Single-Valued Mappings

In this section, we discuss several applications of the random best approximation theorems in Section 2, and give some random fixed point Theorems for single-valued mappings.

THEOREM 3.1. *Let* (Ω, Σ) *be a measurable space with* Σ *a Suslin family and* X *a non-empty Suslin convex subset of a normed space* $(E, \|\cdot\|)$. *Suppose that* $\psi: \Omega \times X \to E$ *is a randomly continuous mapping such that*:

(a) *there exist a non-empty convex compact subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $y \in X \backslash K$, *there exists* $x \in X_0$ *with* $\|x - \psi(\omega, y)\| < \|y - \psi(\omega, y\|$ *for all* $\omega \in \Omega$; *and*

(b) *the function* ψ *satisfies one of the following conditions*:

(i) *for each fixed* $\omega \in \Omega$, *each* $x \in K$ *with* $x \neq \psi(\omega, x)$, *there exists* $y \in I_x(x) := \{x + c(z - x),$ *for some* $z \in X$, *some* $c > 0\}$ *such that* $\|y - \psi(\omega, x)\| < \|x - \psi(\omega, x)\|$.

(ii) ψ *is weakly inward* (*i.e., for each* $\omega \in \Omega$, $\psi(\omega, x) \in \overline{I_X(x)}$ *for* $x \in K$).

Then ψ *has a random fixed point.*

Proof. By Theorem 2.2, there exists a countable measurable family $\{g_i\}_{i=1}^{\infty}$ from Ω to 2^K such that for each g_i,

$$\|g_i(\omega) - \psi(\omega, g_i(\omega))\| = d(\psi(\omega, g_i(\omega)), X)$$

for all $\omega \in \Omega$.

We shall prove that each g_i is a random fixed point of ψ. Let ψ satisfies (i). If there exists some $\omega \in \Omega$ such that $g_i(\omega) \neq \psi(\omega, g_i(\omega))$, then from the assumption (i), there exists $y \in I_x(g_i(\omega))$ such that

$$\|y - \psi(\omega, g_i(\omega))\| < \|g_i(\omega) - \psi(\omega, g_i(\omega))\|.$$

Since $y \in I_X(g_i(\omega))$, there exist $z \in X, c > 0$ such that $y = g_i(\omega) + c(z - g_i(\omega))$, hence $y \notin X$; otherwise we would have a contradiction to the choice of $g_i(\omega)$. Without loss of generality, we assume $c > 1$. Then $z = y/c + (1 - 1/c)\, g_i(\omega) = (1 - \beta)\, y + \beta g_i(\omega)$, where $\beta = 1 - 1/c$, and $0 < \beta < 1$. Hence,

$$\begin{aligned}\|z - \psi(\omega, g_i(\omega))\| &\leqslant (1 - \beta)\, \|y - \psi(\omega, g_i(\omega))\| + \beta\, \|g_i(\omega) - \psi(\omega, g_i(\omega))\| \\ &< (1 - \beta)\, \|g_i(\omega) - \psi(\omega, g_i(\omega))\| \\ &\quad + \beta\, \|g_i(\omega) - \psi(\omega, g_i(\omega))\| \\ &= \|g_i(\omega) - \psi(\omega, g_i(\omega))\|.\end{aligned}$$

This contradicts the choice of $g_i(\omega)$. Therefore $\psi(\omega, g_i(\omega)) = g_i(\omega))$ for each $\omega \in \Omega$, so that g_i is a random fixed point.

If ψ satisfies the condition (ii), for each $\omega \in \Omega$ and each $x \in K$ with $x \neq \psi(\omega, x)$, since $\psi(\omega, x) \in \overline{I_X(x)}$, there exists $y \in I_X(x)$ such that $\|y - \psi(\omega, x)\| < \|x - \psi(\omega, x)\|$, thus ψ satisfies (i). Therefore each g_i is a random fixed point of ψ. ∎

Theorem 3.1 is closely related to, but not comparable to, Theorem 4 of Lin [15], Theorem 4.1 of Tan and Yuan [34] and Theorem 2 of Xu [37].

By Theorem 3.1 we also have the following:

COROLLARY 3.2. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty Suslin convex subset of a normed space $(E, \|\cdot\|)$. Suppose that $\psi: \Omega \times X \to E$ is a randomly continuous mapping. Suppose there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists $x \in X_0$ such that $\|x - \psi(\omega, y)\| < \|y - \psi(\omega, y)\|$ for all $\omega \in \Omega$, and moreover ψ satisfies $f(\omega, \partial K) \subset X$ for each $\omega \in \Omega$. Then ψ has a random fixed point.*

Proof. Since $\psi(\omega, \partial(K)) \subset X$ for each $\omega \in \Omega$ then ψ satisfies the condition (ii) of Theorem 3.1, because $K \subset X \subset I_X(x), I_K(x) \subset I_X(x)$ and $I_K(x) = E$ if $x \in intK$, where ∂K, $intK$ denote the boundary and interior of K in X, respectively. Thus the conclusion follows from Theorem 3.1. ∎

Remark 2. Corollary 3.2 is also closely related to, but not comparable to, the Corollary 1 of Lin [15] and Theorem 3 of Sehgal–Waters [30].

4. RANDOM FIXED POINT THEOREMS FOR NONSELF SET-VALUED MAPPINGS

In the final section of this paper, we study some random fixed point theorems for set-valued mappings as applications of Theorem 2.4 in Section 2.

THEOREM 4.1. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a nonempty Suslin convex subset of a normed space $(E, \|\cdot\|)$. Suppose that $\psi: \Omega \times X \to 2^E$ is a randomly continuous mapping with non-empty compact and convex values such that:*

(a) *there exist a non-empty convex compact subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists $x \in X_0$ with $\inf_{u \in \psi(\omega, y)} \|x - u\| < \inf_{u \in \psi(\omega, y)} \|-u\|$ for each $\omega \in \Omega$; and*

(b) *ψ satisfie one of the following conditions:*

(i) *for each fixed $\omega \in \Omega$, each $x \in K$ with $x \neq \psi(\omega, x)$, there exists $y \in I_X(x) := \{x + c(z - x), \text{for some } z \in X, \text{some } c > 0\}$ such that*

$$\inf_{u \in \psi(\omega, x)} \|y - u\| < \inf_{u \in \psi(\omega, x)} \|x - u\|.$$

(ii) *ψ is weakly inward (i.e., for each $\omega \in \Omega$, $\psi(\omega, x) \cap \overline{I_X(x)} \neq \emptyset$ for each $x \in K$).*

Then ψ has a random fixed point.

Proof. By Theorem 2.4, there exists a countable measurable family $\{g_i\}_{i=1}^{\infty}$ from Ω to K such that for each g_i,

$$\inf_{u \in \psi(\omega, g_i(\omega))} \|g_i(\omega) - u\| = d(\psi(\omega, g_i(\omega)), X)$$

for all $\omega \in \Omega$. Now following the argument of Theorem 3.2, we shall prove that each g_i is a random fixed point of ψ.

Let ψ satisfies (i). If there exists some $\omega \in \Omega$ such that $g_i(\omega) \notin \psi(\omega, g_i(\omega))$, then from assumption (i), there exists $y \in I_X(g_i(\omega))$ such that

$$\inf_{u \in \psi(\omega, g_i(\omega))} \|y - u\| < \inf_{u \in \psi(\omega, g_i(\omega))} \|g_i(\omega) - u\|.$$

Note that $y \in I_X(g_i(\omega))$, so there exist $z \in X, c > 0$ such that $y = g_i(\omega) + c(z - g_i(\omega))$, hence $y \notin X$; otherwise a contradiction to the choice of $g_i(\omega)$ would result. Without loss of generality, we assume that $c > 1$. Then $z := y/c + (1 - 1/c)\, g_i(\omega) = (1 - \beta)\, y + \beta g_i(\omega)$, where $\beta = 1 - 1/c$, and $0 < \beta < 1$. Let $w \in \psi(\omega, g_i(\omega))$ such that $\|g_i(\omega) - w\| = \inf_{u \in \psi(\omega, g_i(\omega))} \|g_i(\omega) - u\| = d(\psi(\omega, g_i(\omega)), X)$. Then we have,

$$\begin{aligned} \|z - w\| &\leqslant (1 - \beta)\, \|y - w\| + \beta\, \|g_i(\omega) - w\| \\ &< (1 - \beta)\, \|g_i(\omega) - w\| + \beta\, \|g_i(\omega) - w\| \\ &= \|g_i(\omega) - w\| = \inf_{u \in \psi(\omega, g_i(\omega))} \|g_i(\omega) - u\| \\ &= d(\psi(\omega, g_i(\omega)), X). \end{aligned}$$

This contradicts the choice of $g_i(\omega)$. Therefore $g_i(\omega) \in \psi(\omega, g_i(\omega))$ for each $\omega \in \Omega$, i.e., g_i is a random fixed point of ψ.

If ψ satisfies (ii), for each $\omega \in \Omega$ and each $x \in K$ with $x \neq \psi(\omega, x)$, there must exist $y \in I_X(x)$ such that $\inf_{u \in \psi(\omega, x)} \|y - u\| < \inf_{u \in \psi(\omega, x)} \|x - u\|$ as $\psi(\omega, x) \cap \overline{I_X(x)} \neq \emptyset$ and ψ is randomly continuous.

Thus, it satisfies the assumption (i). Therefore each g_i is a random fixed point of ψ. ∎

Remark 3. The condition (ii) of Theorem 4.1 was also used by Reich in [21] in the discussing of the existence of random fixed point for non-self randomly continuous condensing mappings.

By the argument similar to that used in the proof of Corollary 3.2, we have the following random fixed point theorem:

COROLLARY 4.2. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a nonempty Suslin convex subset of a normed space $(E, \|\cdot\|)$*

Suppose that $\psi: \Omega \times X \to 2^E$ *is a random continuous mapping with non-empty compact and convex values. If there exist a non-empty compact convex subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that*

(a) *for each* $y \in X \backslash K$, *there exists* $x \in X_0$ *such that* $\inf_{u \in \psi(\omega, y)} \|x-u\| < \inf_{u \in \psi(\omega, y)} \|y-u\|$ *for all* $\omega \in \Omega$; *and*

(b) ψ *satisfies* $\psi(\omega, \partial K) \cap \bar{X} \neq \emptyset$ *for all* $\omega \in \Omega$.

Then ψ *has a random fixed point.*

Proof. Since $\psi(\omega, \partial K) \cap \bar{X} \neq \emptyset$ for all $\omega \in \Omega$, then ψ satisfies the condition (ii) of Theorem 4.1 by the facts that:

(i) $K \subset X \subset I_X(x)$, $I_K(x) \subset I_X(x)$; and

(ii) $I_K(x) = E$ if $x \in intK$, where ∂K, $intK$ denote the boundary and interior of K in X, respectively.

Therefore for each $x \in K$, $\psi(\omega, x) \cap \overline{I_X(x)} \neq \emptyset$ for all $\omega \in \Omega$. Then the conclusion follows from Theorem 4.1. ∎

Remark 4. For the measurable space (Ω, Σ) which has a σ-finite measure, Engl in [7, Theorem 7] gave a stochastic version of the Bohnenblust-Karlin-Kakutani fixed point with the stochastic domain. However his result requires implicitly that the domain X has non-empty interior in a separably Banach space. Moreover, by assuming that the measurable space (Ω, Σ) is complete, Reich in [21] proved a random fixed point for non-self random upper semicontinuous condensing mappings when the underlying space $(E, \|\cdot\|)$ is a Frechet space. Note that all complete measurable spaces and measurable spaces with σ-finite measure have the Suslin family, but the converse is not true in general (e.g., see Rogers [24]), and each closed separable set in a Banach space is a Suslin subset, and the condensing mapping may not satisfy the non-compact condition (ii) of Theorem 4.1. Hence our Theorem 4.1 is independent from those results given by Engl in [7] and Reich in [21]. Furthermore, we note that Theorem 4.1 also generalizes corresponding results of Theorem 2.6 of Tan and Yuan in [33].

So far we have given several random fixed point theorems for set-valued mappings as applications of random variational inequalities. In what follows, we shall consider some random fixed points for random upper semicontinuous mappings which have measurable graphs.

Let X be a topological space with topology T. We shall use (X, T) and $2^{(X, T)}$ to denote spaces X and 2^X respectively with emphasis on the fact that X is equipped with the topology T. Let (E, T) be a topological vector space. For each non-empty subset A of E and for each continuous seminorm p on (E, T), let $d_p(x, A) := \inf_{a \in A} \{p(x-a)\}$. We shall denote by $W := W(E, E^*)$ the weak topology of E. Let X be a non-empty of (E, T).

For each $x \in X$, the inward set and outward set of X at x, denoted by $I_X(x)$ and $O_X(x)$ respectively, are defined by

$$I_X(x) := \{x + r(y - x) : y \in X \text{ and } r > 0\},$$

$$O_X(x) := \{x - r(y - x) : y \in X \text{ and } r > 0\}.$$

The closures of $I_X(x)$ and $O_X(x)$ in (E, T), denoted by $\overline{I_X(x)}$ and $\overline{O_X(x)}$ respectively, are called the weakly inward and the weakly outward set of X.

As a simple application of Theorem 2.3 of Tan and Yuan [33], we have the following random fixed point theorems for non-self upper semicontinuous mappings with measurable graphs in locally convex spaces. These results also improve corresponding results of Theorem 2.6 of Tan and Yuan [33] in several aspects.

THEOREM 4.3. *Let (Ω, Σ) be a measurable space with Σ a Suslin family. Let (E, T) a locally convex Hausdorff topological vector space, X a non-empty Suslin, W-compact convex subset of E and $F: \Omega \times (X, W) \to 2^{(E, T)}$ be random upper semicontinuous with non-empty T-compact convex values and Graph $F \in \Sigma \otimes \mathscr{B}((X, W) \times (E, T))$. For each fixed $(\omega, x) \in \Omega \times X$, if each weakly continuous semi-norm p on E with $d_p(x, F(\omega, x)) > 0$, we have for any $u \in F(\omega, x)$,*

$$d_p(u, \overline{I_X(x)})) < p(x - u) \ (resp., \ d_p(u, \overline{O_X(x)})) < p(x - u).$$

Then F has a random fixed point.

Proof. For each fixed $\omega \in \Omega$, the mapping $F(\omega, \cdot): (X, W) \to 2^{(E, T)}$ satisfies all hypotheses of Theorem 5 of Ding and Tan [6, p. 795]. Hence $F(\omega, \cdot)$ has a fixed point in (X, T). Then the conclusion follows by Theorem 2.3 of Tan and Yuan [33]. ∎

Theorem 4.3 is a random version of Theorem 5 of Ding and Tan [6] which in turn improves Theorem 2 of Reich [23]. From the proof of Theorem 4.3 and by combining Theorem 6 of Ding and Tan [6], we have the following:

THEOREM 4.4. *Let (Ω, Σ) be a measurable space with Σ a Suslin family. Let (E, T) be a locally convex Hausdorff topological vector space, X be a non-empty Suslin, W-compact convex subset of E, and $F: \Omega \times (X, W) \to 2^{(E, T)}$ be random upper semicontinuous with non-empty T-compact convex values and Graph $F \in \Sigma \otimes ((X, W) \times (E, T))$. Suppose that for each fixed $\omega \in \Omega$, $x \in \partial_{(E, W)} X \setminus F(\omega, x)$ and for each $u \in F(\omega, x)$, there* *exists a number λ (real or complex, depending on whether the vector space E is real or complex) with $|\lambda| < 1$ such that*

$$\lambda x + (1-\lambda)u \in \overline{I_X(x)} \text{ (respectively } \lambda x + (1-\lambda)u \in \overline{O_X(x)}),$$

here $\partial_{(E, W)} X$ denotes the boundary of X in the topological space (E, W).

Then F has a random fixed point.

Proof. For each fixed $\omega \in \Omega$ the mapping $F(\omega, \cdot)$: $(X, W) \to 2^{(E, T)}$ satisfies all hypotheses of Theorem 6 of Ding and Tan [6, p. 795]. Hence $F(\omega, \cdot)$ has a fixed point in (X, T). Then the conclusion follows by Theorem 2.3 of Tan and Yuan [33]. ∎

Theorem 4.4 is a random version of Theorem 6 of Ding and Tan [6] which in turn improves Theorem 3.1 of Reich [22] and Theorem 3 of Fan [8].

Remark 5. Let (Ω, Σ) be a measurable space and X and Y be topological spaces. Let F: $\Omega \times X \to 2^Y$ be a random continuous mapping with non-empty closed values. When X and Y are both separable metric spaces, it is clear that F has a measurable graph by Lemma 2.4 in [33]. In this case, Corollary 4.2 can be derived by Theorem 4.3 or Theorem 4.4. However neither X nor Y is a separable metric space, we can not prove that the randomly continuous mapping F with non-empty closed values has a measurable graph (for more details, see Himmelberg [10], Kucia and Nowak [12], Wanger [36] and reference therein). Therefore both Theorems 4.3 and 4.4 can not include Corollary 4.2 (so that theorem 4.1) as a special case if the Suslin space X is not metrizable.

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